From Quantum Entanglement to Model Reduction

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Smoothness and Sparsity

PDE model: $u \in \mathcal{V}$

$$F(u, Du, \dots, D^n u, x) = 0, \quad x \in \mathbb{R}^d.$$

Numerical approximation: $u_h \in \mathcal{V}_h$

$$F_h(u_h, D_h u, \dots, D_h^n u, x) = 0, \quad x \in \mathbb{R}^d.$$

Optimal rates for $\|u - u_h\|_{\mathcal{V}}$?

$\mathcal{V} \in \{B^s, W^s, ...\}$, $\mathcal{V}_h \in \{$splines, wavelets, ...$\}$ $\Leftrightarrow^1$ $\|u - u_h\|_{\mathcal{V}} = \mathcal{O}(h^{s/d})$.

Low-Rank approximation:

$$u \approx u_r = \sum_{k=1}^{r} v_k^1 \otimes \cdots \otimes v_k^d.$$

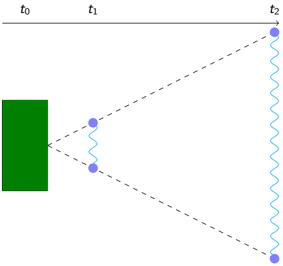
$\mathcal{V} \in \{B^s, W^s, ...\}$ $\Rightarrow^2$ $\|u - u_h\|_{\mathcal{V}} = \mathcal{O}(r^{-s/d})$. ↯

[1] DeVore: *Nonlinear Approximation* (1998).

[2] Schneider, Uschmajew: *Approximation Rates for the Hierarchical Tensor Format in Periodic Sobolev Spaces* (2014).

Quantum Entanglement

First mention in EPR[3] (EPR Paradox).



Mathematical description: $\psi \in \mathcal{V}_1 \otimes \mathcal{V}_2$

$$\psi = \sum_{k=1}^{\infty} \sigma_k v_k \otimes w_k, \quad |\{k \in \mathbb{N} : \sigma_k > 0\}| > 1.$$

[3] Einstein, Podolsky, Rosen: *Can Quantum-Mechanical Description of Physical Reality be Considered Complete?* (1935).

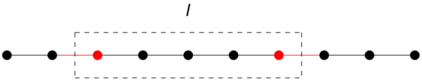
Entropy

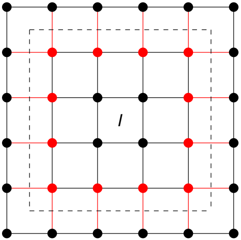
Quantifying information: state $\rho \in \mathcal{T}(\mathcal{V})$,

$$S(\rho) = \mathrm{tr}[\rho \log \rho] = \mathbb{E}[\log \rho] = \sum_{k=1}^{\infty} \lambda_k \log(\lambda_k) \quad (\textit{Von-Neumann Entropy}).$$

Area Laws[4]:

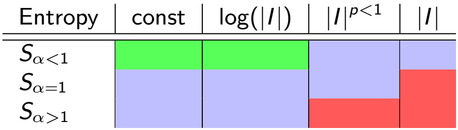
$$S(\rho_I) = \mathcal{O}(|\partial I|).$$



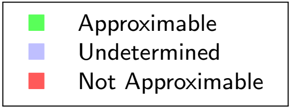


[4] Eisert, Cramer, Plenio: *Area Laws for the Entanglement Entropy - a Review* (2010).

Approximability



1D Chains[5]



[5] Schuch, Wolf, Verstraete, Cirac: *Entropy Scaling and Simulability by Matrix Product States* (2008).

Local Interaction Systems

Mission

Operator structure $\Rightarrow$ Area Law in continuous systems

NNI systems:

$$H = \sum_{I \subset X} H_I .$$

Properties:

- gap,
- local support of H_I,
- finite interaction strength.

Outlook: long-range interactions, exponentially decaying correlations.

Importance of Spectrum

Can show[6]

Theorem

f approximable with rate α and $Bu = f$ (separable B) $\Rightarrow$ u approximable with rate $\tilde{\alpha} \approx \alpha$.

Ingredients:

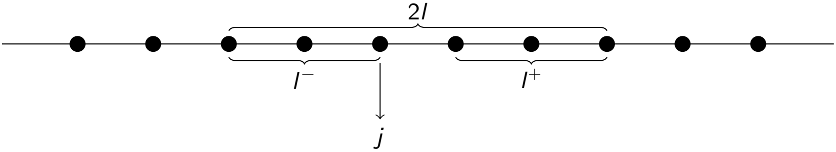
- Separable eigenvectors B $\Rightarrow$ separable eigenvectors B^{-1}.
- Exponential sum approximation of $\sigma(B^{-1})$.
- Constant depends only on coercivity ($\sim$ gap).

Conjecture

Spectrum separability $\Rightarrow$ separability of general solutions.

[6] Dahmen, DeVore, Grasedyck, Süli: *Tensor-Sparsity of Solutions of High-Dimensional Elliptic Partial Differential Equations* (2015).

Area Law 1D Spin Systems[8]



$$P_0 \approx O_B(2l)O_L(1,j)O_R(j+1,d).$$

By Lindblad-Uhlmann[7]

$$S(2l||l^- \otimes l^+) \geq S(O_B[2l]||O_B[l^- \otimes l^+]) \geq \mathbb{E}[O_B] + S(1,j).$$

$$\Rightarrow 0 \leq S(2l) \leq 2S_l - \mathbb{E}[O_B] - S(1,j) \leq [(\max\ S)l - \log(l)l] - S(1,j).$$

[7] Lindblad: *Completely Positive Maps and Entropy Inequalities* (1974).

[8] Hastings: *An Area Law for One Dimensional Quantum Systems* (2018).

Finite Entropy

- *Most* states have infinite entropy [9].
- *Physically meaningful* states have finite entropy.

Lemma (Decay Rate)

If $\sigma_k^l(\psi) \lesssim k^{-p}$ *for* $p > 1/2$*, then* $S(\rho_l) < \infty$.

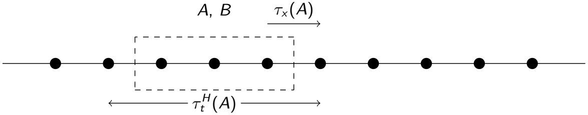
Lemma (Finite Energy Gibbs State)

If $E := \mathrm{tr}(\rho H) < \infty$ *and* $\exp(-\beta H) \in \mathcal{T}(V)$ *for some* $\beta > 0$*, then* $S(\rho_l) < \infty$.

[9] Wehrl: *General Properties of Entropy* (1978).

Finite Speed Propagation

Lieb-Robinson bound(s) [10]



$$\lim_{\substack{|t|\to\infty,\\ |x|>v|t|}} \left\| [\tau_t^H \tau_x(A), B] \right\| \exp(c|t|) = 0.$$

$$H = H_L + H_B + H_R,\ \tilde{H}^0_{L/B/R} = \int_{-\infty}^{\infty} \tau_t^H \left(H_{L/B/R}\right) W_I(t)dt,\ \tilde{H}^0_{L/B/R}[\psi_0] \approx 0.$$

Lemma/Conjecture

If H is local with finite interaction strength, then $\exists\ O_L,\ O_B,\ O_R$

$$\|O_B O_L O_R - P_0\| \lesssim \exp(-cl).$$

[10] Lieb, Robinson: *The Finite Group Velocity of Quantum Spin Systems* (1972).

Summary and Outlook

- Entropy measures promising for quantifying reducibility.
- Area Laws describe systems breaking c.o.d.
- Many challenges remain:
  - Area Laws for continuous systems.
  - The right entropy measure.
  - Multi-dimensional Area Laws/Non-MPS systems.

“Before I came here I was confused about this subject. Having listened to your lecture I am still confused. But on a higher level.“

Attributed to Enrico Fermi (29 Sep. 1901 – 28 Nov. 1954)